

## “PHASING” OPERATOR FOR TWO OSCILLATORS IN CLASSICAL FIELD

Jong-Jean Kim and Je-Hwan Koo

*Physics Department, Korea Advanced Institute of Science and Technology,  
371-1 Kusong-dong, Yusung-ku, Taejeon 305-701, Korea*

Dong-Jae Bae

*Physics Department, Pohang Institute of Science and Technology  
P.O.Box 125, Pohang 790-600, Korea*

### Abstract

The origin of the Dicke cooperative states, ad hoc assumed for superradiance in the system of molecules where no mutual interactions exist but all encountering the same field of radiation, was studied by considering two harmonic oscillators driven by a common field of radiation. A phasing operator as  $\Phi_N = D(\alpha)^+ P_N D(\alpha)$ , where  $D(\alpha)$  is the displacing operator and  $P_N$  the projection operator for constant energy  $N$  for two oscillators, was derived. The eigen states of the phasing operator  $\Phi_N$  are found to show a finite correlation as for the Dicke cooperative states.

## 1 Introduction

The important notion put forward by Dicke [1], that molecules can not be treated as independent when the molecules are interacting each with a common field of radiation, introduced the well known phenomena of superradiance with the ad hoc assumption of cooperative states. This ad hoc assumption of the Dicke's cooperative states may be a natural consequence of the fact that a forced quantum oscillator can be described in terms of the Glauber's coherent states [2]. However, it may be more enlightening to examine in rigorous quantum mechanics how two independent molecules (harmonic oscillators) are getting correlated simply by having separate interactions with a common field of radiation.

Furthermore, it may be more interesting if we restrict the interaction between the radiation and the molecular system to be “elastic”, that is, no net transfer of energy between the molecular system and the radiation field.

Naively this restriction may correspond to an elastic light scattering from the two molecule system and a possibility of phasing or correlation of the two molecules by this continuous scattering of light ( more correctly a driving field of radiation in the elastic channel of interaction).

## 2 Two Driven Oscillators

We consider two harmonic oscillators driven by a classical field :

$$H = \hbar\omega(a_1^\dagger a_1 + a_2^\dagger a_2) - x_0(a_1^\dagger + a_1 + a_2^\dagger + a_2)E(t) \quad (1)$$

where we do not have a direct coupling between the two oscillators. This system seems to be considered as trivial because a single harmonic oscillator driven by a classical field is fully understood [3].

Since we will be extending the solutions of the single harmonic oscillator driven by a classical field to the two oscillator problem of Eq.(1), we want to recollect here some important results of a driven harmonic oscillator [3] :

$$D(\alpha)|0\rangle = \exp(\alpha a^\dagger - \alpha^* a)|0\rangle = |\alpha\rangle \quad (2)$$

$$D(\alpha)|n\rangle = ((a^\dagger - \alpha^*)^n / \sqrt{n!})|\alpha\rangle \quad (3)$$

where  $D(\alpha)$  is the displacement operator,  $|\alpha\rangle$  coherent state of Glauber,

$$\alpha = (ix_0/\hbar) \int_{-\infty}^{\infty} E(t) \exp i\omega t dt, \quad \exp(\alpha a^\dagger - \alpha^* a) = \exp(-|\alpha|^2/2) \exp(\alpha a^\dagger) \exp(-\alpha^* a).$$

We address now to the two oscillator problem of Eq.(1).

Suppose the two oscillators are prepared in a state  $|N\rangle$  of total energy  $N = n_1 + n_2$ , then we let  $|N\rangle$  be driven by a classical field  $D(\alpha)$  but we restrict the driven system  $D(\alpha)|N\rangle$  to remain at the same energy of  $|N\rangle$ .

The quantum mechanical matrix element corresponding to this process may be written as

$$\langle N|D^\dagger(\alpha)P_N D(\alpha)|N\rangle \quad (4)$$

where  $P_N$  represents the projection operator for states of total energy  $N$  :

$$P_N = \sum_{n=0}^N |N-n\rangle\langle n| \langle n| \langle N-n| \quad (5)$$

Alternatively we may define a new operator :

$$\Phi_N \equiv D^\dagger(\alpha)P_N D(\alpha) \quad (6)$$

and Eq.(4) can be written as

$$\langle N|\Phi_N|N\rangle \quad (7)$$

This implies some particular  $|N\rangle$  states can become eigen states of the operator  $\Phi_N$ .

### 3 Evaluation of $\langle N | \Phi_N | N \rangle$

For two harmonic oscillator states of total energy  $N$  we can write

$$|N\rangle = \sum_{n=0}^N c_n |N-n\rangle |n\rangle \quad (8)$$

From eq.(3) and eq.(8) we obtain

$$D(\alpha)|N\rangle = \sum_{n=0}^N c_n \left( (a_1^+ - \alpha_1^*)^{N-n} (a_2^+ - \alpha_2^*)^n / \sqrt{(N-n)!} \sqrt{n!} \right) |\alpha\rangle_{(1)} |\alpha\rangle_{(2)} \quad (9)$$

where the suffix (1) and (2) refer to the oscillator index. Equation (5) can be rewritten as

$$P_N = \sum_{n=0}^N \left( (a_1^+)^{N-n} (a_2^+)^n / \sqrt{(N-n)!} \sqrt{n!} \right) |0\rangle_{(2)} |0\rangle_{(1)} \langle 0|_{(1)} \langle 0|_{(2)} \left( a_2^n a_1^{N-n} / \sqrt{n!} \sqrt{(N-n)!} \right) \quad (10)$$

From eqs.(6), (9) and (10) we obtain

$$\langle N | \Phi_N | N \rangle = \sum_{n=0}^N \left| \sum_{m=0}^N c_m \langle 0|_{(2)} \langle 0|_{(1)} \{ a_2^n (a_2^+ - \alpha_2^*)^m a_1^{N-n} (a_1^+ - \alpha_1^*)^{N-m} / \sqrt{n!} \sqrt{m!} \sqrt{(N-n)!} \sqrt{(N-m)!} \} |\alpha\rangle_{(1)} |\alpha\rangle_{(2)} \right|^2 \quad (11)$$

Making use of  $a|\alpha\rangle = \alpha|\alpha\rangle$  and  $a^n(a^+ - \alpha^*)^m = (a^+ - \alpha^*)^m a^n + m!(a^+ - \alpha^*)^{m-n}/(m-n)!$  we can obtain

$$\begin{aligned} \langle N | \Phi_N | N \rangle &= \sum_{n=0}^N \left| \sum_{m=0}^N c_m \langle 0|_{(2)} \langle 0|_{(1)} \{ \sqrt{m!} (a_2^+ - \alpha_2^*)^{m-n} / \sqrt{n!} (m-n)! \right. \\ &\quad + (a_2^+ - \alpha_2^*)^m \alpha_2^n / \sqrt{m!} \sqrt{n!} \} \{ \sqrt{(N-m)!} (a_1^+ - \alpha_1^*)^{n-m} / \sqrt{(N-n)!} (n-m)! \\ &\quad \left. + (a_1^+ - \alpha_1^*)^{N-m} \alpha_1^{N-n} / \sqrt{(N-m)!} \sqrt{(N-n)!} \} |\alpha\rangle_{(1)} |\alpha\rangle_{(2)} \right|^2 \end{aligned} \quad (12)$$

Since we are dealing with two identical molecules and the same common field of radiation we may set  $\alpha_1 = \alpha_2 = \alpha$  and make use of the following [3],

$$a|0\rangle = 0, \quad \langle 0|\alpha\rangle = \exp(-|\alpha|^2/2) \quad (13)$$

to obtain

$$\begin{aligned} \langle N | \Phi_N | N \rangle &= \sum_{n=0}^N \sum_{m=0}^N \sum_{l=0}^N c_m c_l^* \{ \exp(-2|\alpha|^2)/n!(N-n)! \} \{ \sqrt{m!} \sqrt{(N-m)!} \delta_{mn} \\ &\quad + (-1)^{N-n} \sqrt{m!} |\alpha|^{2(N-n)} / (m-n)! \sqrt{(N-m)!} \\ &\quad + (-1)^n \sqrt{(N-m)!} |\alpha|^{2n} / (n-m)! \sqrt{m!} \\ &\quad + (-1)^N |\alpha|^{2N} / \sqrt{m!} \sqrt{(N-m)!} \} \{ \sqrt{(N-l)!} \sqrt{l!} \delta_{ln} \\ &\quad + (-1)^{N-n} \sqrt{l!} |\alpha|^{2(N-n)} / (l-n)! \sqrt{(N-l)!} \\ &\quad + (-1)^n \sqrt{(N-l)!} |\alpha|^{2n} / (n-l)! \sqrt{l!} \\ &\quad + (-1)^N |\alpha|^{2N} / \sqrt{l!} \sqrt{(N-l)!} \} \end{aligned} \quad (14)$$

Since  $|N\rangle$  of eq.(8) can be represented by a  $(N+1)$ -dimensional state vector,  $\Phi_N$  can be defined as a  $(N+1) \times (N+1)$  matrix operator to give

$$\langle N | \Phi_N | N \rangle = \sum_{m=0}^N \sum_{l=0}^N c_m c_l^* \Phi_N^{lm} \quad (15)$$

where we find, from eq.(14), the  $(l,m)$  matrix element of the operator  $\Phi_N$  as follows:

$$\begin{aligned} \Phi_N^{lm} = & \sum_{n=0}^N (\exp -2|\alpha|^2/n!(N-n)!) \{ \sqrt{m!} \sqrt{(N-m)!} \delta_{mn} \\ & + (-1)^{(N-n)} \sqrt{m!} |\alpha|^{2(N-n)} / (m-n)! \sqrt{(N-m)!} \\ & + (-1)^n \sqrt{(N-m)!} |\alpha|^{2n} / (n-m)! \sqrt{m!} \\ & + (-1)^N |\alpha|^{2N} / \sqrt{m!} \sqrt{(N-m)!} \} \{ \sqrt{(N-l)!} \sqrt{l!} \delta_{ln} \\ & + (-1)^{N-n} \sqrt{l!} |\alpha|^{2(N-n)} / (l-n)! \sqrt{(N-l)!} \\ & + (-1)^n \sqrt{(N-l)!} |\alpha|^{2n} / (n-l)! \sqrt{l!} \\ & + (-1)^N |\alpha|^{2N} / \sqrt{l!} \sqrt{(N-l)!} \} \end{aligned} \quad (16)$$

We can check for the correct limiting values:

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \Phi_N^{lm}(\alpha) &= \delta_{lm} \\ \lim_{\alpha \rightarrow 0} \langle N | \Phi_N | N \rangle &= \sum_{n=0}^N \left| \sum_{m=0}^N c_m \left( \sqrt{m!} \sqrt{(N-m)!} / \sqrt{n!} \sqrt{(N-n)!} \right) \delta_{mn} \right|^2 \\ &= \sum_{n=0}^N |c_n|^2 = 1 \end{aligned}$$

## 4 $\Phi_N$ As Phasing Operator

In order to probe into the physical meaning of the operator  $\Phi_N$  we illustrate for the simplest nontrivial case of  $N = n_1 + n_2 = 1$ .

The  $\Phi_N$  operator is then given in the form

$$\Phi_{N=1} = \begin{pmatrix} \Phi_1^{00} & \Phi_1^{01} \\ \Phi_1^{10} & \Phi_1^{11} \end{pmatrix} \quad (17)$$

and the matrix elements are obtained from eq.(16) as follows:

$$\begin{aligned} \Phi_1^{00} &= \Phi_1^{11} = \exp(-2|\alpha|^2)(4 - 8|\alpha|^2 + 8|\alpha|^4) \\ \Phi_1^{01} &= \Phi_1^{10} = \exp(-2|\alpha|^2)(-8|\alpha|^2 + 8|\alpha|^4) \end{aligned}$$

Diagonalizing the matrix of eq.(17) gives the eigen values  $\lambda_1$  and  $\lambda_2$  as

$$\begin{aligned}\lambda_1 &= \Phi^{00} + \Phi^{01}, \\ \lambda_2 &= \Phi^{00} - \Phi^{01}\end{aligned}\tag{18}$$

The corresponding eigen states  $|\lambda_1\rangle$  and  $|\lambda_2\rangle$  are obtained as follows:

$$\begin{aligned}|\lambda_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle_{(1)}|1\rangle_{(2)} + |1\rangle_{(1)}|0\rangle_{(2)}), \\ |\lambda_2\rangle &= \frac{1}{\sqrt{2}}(|0\rangle_{(1)}|1\rangle_{(2)} - |1\rangle_{(1)}|0\rangle_{(2)})\end{aligned}\tag{19}$$

Correlation or phasing of two oscillators can be measured by a value of  $\langle x_1 \cdot x_2 \rangle$  where  $x_1$  and  $x_2$  are the two harmonic oscillator coordinates of displacement [4].

We can see easily

$$\langle 0|_{(2)}\langle 1|_{(1)}x_1 \cdot x_2|1\rangle_{(1)}|0\rangle_{(2)} = \langle 0|_{(1)}\langle 1|_{(2)}x_1 \cdot x_2|1\rangle_{(2)}|0\rangle_{(1)} = 0$$

for the case of  $\alpha = 0$ , but for our eigen states of  $\Phi_N$  we obtain

$$\begin{aligned}\langle \lambda_1|x_1 \cdot x_2|\lambda_1\rangle &= \langle \lambda_2|x_1 \cdot x_2|\lambda_2\rangle \\ &= \frac{1}{2}x_0^2 \{ \langle 0|_{(2)}\langle 1|_{(1)}a_1^+a_2|0\rangle_{(1)}|1\rangle_{(2)} + \langle 1|_{(2)}\langle 0|_{(1)}a_1a_2^+|1\rangle_{(1)}|0\rangle_{(2)} \} \\ &= x_0^2\end{aligned}$$

We can thus see that the two noninteracting molecules in the common driving field of radiation find themselves as correlated. The correlated eigen states of the concerning Hamiltonian of eq.(1) can be found as the eigen states of the phasing operator  $\Phi_N$  as introduced in eq.(6).

The existence of the correlated eigen states of the two oscillator Hamiltonian thus justifies the ad hoc assumption of the Dicke cooperative states for independent molecules all in the same common field of radiation.

## 5 Acknowledgements

This work was supported in part by the Korea Science and Engineering Foundation (Center for Thermal and Statistical Physics). One of the authors(JJK) is grateful to Prof. V. I. Manko(Lebedev Institute of Physics) and Prof. Y. S. Kim(University of Maryland) for their kind invitation to the Workshop, and also to Prof. David T. Phillips for his suggestion of this problem at University of California, Santa Barbara in 1968.

## References

- [1] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
- [2] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).  
P. Carruthers and M. Nieto, Am. J. Phys. **33**, 537 (1965).
- [3] W. H. Louisell, *Quantum Statistical Properties of Radiation*, John Wiley & Sons, New York (1973).
- [4] G. W. Parker, Am. J. Phys. **44**, 394 (1976).